Fundamentals of Special Relativity

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Abstract

In this document, we first examine the fundamental ideas of the special theory of relativity, namely the principle of relativity, the law of propagation of light in vacuum and the experiments that inspired physicists like Einstein and Lorentz to develop this theory. We will then motivate and derive the Lorentz transformation and outline its importance in special relativity. To conclude we explicitly compute the isometries of the Minkowski spacetime (the "flat" spacetime of special relativity), and demonstrate that the Lorentz transformations preserve Minkowski spacetime distances.

1 Fundamental Concepts

The Principle of Relativity

To make precise statements about space and time, one needs to refer to some coordinate system (or "frame"). Time intervals and space intervals are measured with respect to this frame, using some physical measurement device (for example, a metre stick for measuring distance and the decay of a sample of Caesium atoms for measuring time). We will deal with a special class of such coordinate systems called Inertial frames, the defining characteristic of which is that a body which is experiencing no net force will be measured in that frame as being stationary or as moving in a straight line with uniform velocity. One can immediately see the connection to Newtons third law, as Newtonian mechanics was derived implicitly in the language of inertial frames, building on ideas originally postulated by Galileo .

The importance of choosing an inertial frame becomes a bit clearer with an example. A very accessible example of a non-inertial frame is that of an observer standing on a steadily rotating disk (and thus, experiencing a constant acceleration toward the

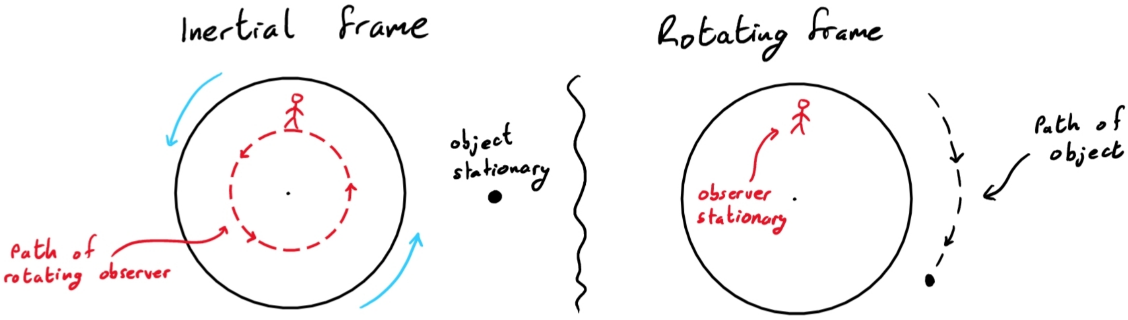

Figure 1: A demonstration of the apparent accelerated motion of an object with no net force acting on it as a result of using a non-inertial frame. For the purpose of the drawing, the inertial frame is chosen to be at rest with respect to the object.

center of the disk, only being held in place by the friction of their shoes), who measures distance with their metre stick and time with whatever clock they choose. From the perspective of this observer, a body just outside the disk with no net force acting on it appears to be spinning around them in a large circle. Thus the laws of physics don't look the same in this frame as in an inertial frame outside the disk, wherein the body would either be stationary or moving uniformly in a straight line. Here we have seen a simple example of the case of "fictitious forces" such as the Coriolis and centrifugal force, which arise in Newtonian mechanics when we don't use inertial frames.

The principle of relativity is the following assumption about the nature of physical laws: *The laws of physics are the same in all inertial frames.* This assumption is a very powerful one, but appears to be correct as no experiment has ever contradicted it. This principle was first explicitly postulated by Galileo, where he argued that a person under the deck of a ship would experience no difference in experiment (eg. throwing an object or the flight paths of flies) whether the ship was stationary or moving smoothly and uniformly, independent of the speed of the ship. Newton used this idea to formulate explicit mathematical laws of motion in his Principia Mathematica [1], along with an idea of "absolute time" which will soon prove to be problematic[1].

The Law of Propagation of Light

The law of propagation of light in vacuum is as follows: *All light travels through vacuum at a constant speed* $c = 2.998... \times 10^8$ ms^{-1}. To come upon such a law, we answer three famous questions (in this section, "speed" is as measured in an inertial frame):

[1]This is essentially the statement that simultaneity of events is preserved across all inertial frames. This is a statement which cannot be true when we acknowledge the law of propagation of light in vacuum.

1. **Can this propagation speed depend on the frequency of the light?**
   There is a wonderful argument that in fact, the speed cannot vary with frequency (or equivalently, the wavelength or energy). Consider a solar eclipse in which the moon blocks out our sun. If for example blue light travelled faster than red, then as the moon covered the sun we should first see a pulse of intensity of blue light, followed by the same for red. This would be due to the fact that when the moon is blocking the light of the sun, the blue light between the Earth and the moon would travel to the surface faster than the red. If everything happened on human-perceivable timescales, we would see a very blue and then very red solar eclipse. This extends outside the visible spectrum and to tiny timescales via sophisticated imaging equipment, and while there are some colour distortions due to partial coverage of the Sun's chromosphere, this argument has been proven to hold within detectable limits.

2. **Can the propagation speed change if the source is moving?**
   The answer to this question is also no, and the proof[2] was provided by De Sitter in 1913. The argument goes as follows: Consider a star orbiting around the center of mass of a double-star system, with an orbital plane not perpendicular to our line of view. If the speed of the star were to influence the speed of the light it emits, there would be a periodic variation of the speed of light in the direction of Earth. If this effect were large, the light we observe from double star systems would exhibit very erratic behaviour, and it would not be possible to derive predictions such as Kepler's laws of motion, which even back in 1913 had been shown to hold very accurately for the motions of planets and stars. This does leave the possibility that there is a very small effect, but this would vary with distance to the double star system in such a way that has never been observed within detectable limits. Note that this point also means motion of an observer relative to a source cannot change the measurement of the speed of light, by a symmetrical argument.

3. **Can motion of the propagation medium change the speed?**
   Many experiments attempted to answer this question in the 19th and 20th centuries, one of which was conducted by Fizeau in 1851 when he measured the speed of light propagating through a liquid which moves with uniform velocity in a tube. Accounting for the refractive index of the medium[3] he found that indeed the light did appear to be "dragged along" by the medium when measured in the lab frame but that this effect was far less than what would be expected by simply adding the (retarded) velocity of the light to that of the liquid. However to answer our

[2] In fact, experimental evidence did not support De Sitter's theory until more sophisticated methods were used by Brecher 64 years later [3]

[3] A phenomenon which can be accounted for by studying Maxwell's equations at a boundary, and results in an apparent "slowing down" of light in media, depending on the dielectric properties of that medium.

question about the law of propagation of light in vacuum, it was also shown that if the medium had index of refraction 1 (as vacuum does) the speed of the "moving" light would be exactly c. This "dragging" effect observed when the medium has index of refraction different from 1 will be explained by a new way to add velocities, which we will see in the next section.

Note that these claims have since been verified by many experiments, which have grown in sophistication and accuracy over time such that we now have little choice but to accept that this law indeed holds. We now discover the apparently fatal incompatibility of this law with the principle of relativity, by examining how we add velocities and the problems that can arise in doing so. Luckily (as is suggested by "apparently") this incompatibility will prove to be nonexistent, allowing us to obtain an elegant and simple theory of time and space, based on very few assumptions.

2 The Lorentz Transformation

To be faithful to Einstein's train analogies (as in [2]), we consider two frames K and K', described by coordinates (t, x, y, z) and (t', x', y', z'), respectively. The first one corresponds to an observer standing on a railway embankment, considered a static frame, and the latter corresponds to an observer who is in a train moving at constant speed v, in the x direction. Both reference systems are inertial systems, i.e. Newtons first law holds and according to the principle of relativity, we also have that Newtons second law holds. The relation between the coordinates of the two systems, classically, is given by the Galilean transformation:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t, \tag{1}$$

as a direct consequence from (1), the velocities u_x and u'_x with respect to the frames K and K', respectively, satisfy

$$u_x = u'_x + v. \tag{2}$$

This is an intuitive result; a high school student would know that if someone inside the train threw a ball with a speed u'_x (measured inside the train) and the train is moving at a speed v in the same direction, someone who is "static" outside the train would see the ball moving at a speed $u_x = u'_x + v$, i.e. they would simply add the velocities. The problem would arise if we replace the ball by a flashlight and we were asked to compute the speed of a photon (emitted from it) from the perspective of an observer on the embankment. If (2) were true, the photon would have different speeds for each frame. However, this contradicts the fact of nature that the speed of light in vacuum (we consider the train example as being in vacuum, neglecting the air) is the same for all inertial systems. Therefore, the Galilean transformation (2) cannot hold nor can the change of coordinates (1). Therefore we need a change of coordinates for the frames K and K' which is compatible with the theory of special relativity, and such transformations between coordinates are called Lorentz Transformations.

Derivation of the Lorentz Transformation

The key concept involved in obtaining a transformation which is indeed compatible with the law of propagation of light in vacuum and the principle of relativity, is to abandon the assumptions of absolute space and time from Newtonian mechanics. That is, we can no longer assume that a space or time interval measured in one inertial frame will be the same as that measured in another inertial frame (this is the advent of the concepts of time dilation and length contraction). Let us now derive the Lorentz transformation as was originally presented by Einstein [2].

Consider a reference frame K described with coordinates (t, x, y, z), and a reference frame K' with coordinates (t', x', y', z') moving with a constant velocity v along the x axis with respect to K, such that the x and x' axes permanently coincide. We derive[4] the transformations between the primed and unprimed coordinates by first explicitly imposing the law of propagation of light in vacuum for a photon[5] travelling along the positive x axis:

$$\begin{aligned} x &= ct, \\ x' &= ct'. \end{aligned} \tag{3}$$

If $(x - ct) = 0$ and $(x' - ct') = 0$ is always satisfied for the photon, then we must have

$$x' - ct' = \lambda(x - ct) \tag{4}$$

for some constant λ. The same consideration for a light ray travelling along the negative x axis yields

$$x' + ct' = \mu(x + ct) \tag{5}$$

for some constant μ. By both adding and subtracting equations 4 and 5, we obtain

$$\begin{aligned} x' &= \gamma x - bct, \\ ct' &= \gamma ct - bx, \end{aligned} \tag{6}$$

where we have (suggestively) defined the constants

$$\begin{aligned} \gamma &:= (\lambda + \mu)/2, \\ b &:= (\lambda - \mu)/2. \end{aligned} \tag{7}$$

[4]Note that the method presented here is one of many equivalent derivations based on the same principles, and which all yield identical results.

[5]Here, referring to a photon is entirely equivalent to just tracking a specific part of the light ray in a consistent manner, i.e. quantum mechanics plays no role.

To fix b in terms of γ, we consider the origin of K' for which $x'_{\text{origin}} = 0$ and $x_{\text{origin}} = vt$ always holds. Comparison with the first part of eq. 6 when $x' = 0$ shows us that

$$b = \frac{v}{c}\gamma. \tag{8}$$

Therefore the Lorentz transformation in terms of γ is given by

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right), \end{aligned} \tag{9}$$

where it is important to note that the y and z coordinates are invariant, ie $y' = y, z' = z$. Finally to fix γ we consider the symmetry of the principle of relativity: the length (as measured by K) of a unit measuring rod at rest relative to K' must be the same as the length (as measured by K') of a unit measuring rod at rest relative to K. This must hold at all times, so we can consider the case of $t = 0$. When $t = 0$ we have from eq. 6 that $x' = \gamma x$, and therefore a unit measuring rod in K' with $\Delta x' = 1$ will be measured in K as

$$\Delta x = \Delta x'/\gamma = 1/\gamma. \tag{10}$$

By considering the same situation but with $t' = 0$ and a unit measuring rod in K with $\Delta x = 1$ we obtain

$$\Delta x' = \gamma\left(1 - \frac{v^2}{c^2}\right)\Delta x = \gamma\left(1 - \frac{v^2}{c^2}\right). \tag{11}$$

Thus exploiting this symmetry and equating these two length measurements we find that the so-called "Lorentz factor" is given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{12}$$

Now looking at the Lorentz transformation and noting that $\gamma > 1$, the origins of length contraction and time dilation become clear; length intervals measured at rest with respect to events will always be longer than those measured by inertial frames moving with respect to said events. We refer to "length contraction" because an observer moving quickly past an object will measure that object to be shorter than an observer holding the object would[6]. Similarly, in a compensatory fashion so as to keep the speed of light constant, time intervals measured at rest with respect to events will always be shorter than those measured by inertial frames moving with respect to these events.

[6] And symmetrically, the observer with the object would see a very compressed-looking observer fly past them, where the compression takes place only in the direction of motion.

3 Isometries of the Minkowski Spacetime

Let us find a characterization of the isometries in the Minkowski spacetime. We will work in arbitrary dimension, namely on $(\mathbb{R}^{1,n}, \eta)$ where η is the Minkowski metric in n+1 dimensions diag(-1,1,....,1).

Recall that an isometry is defined as a distance-preserving diffeomorphism. In this section, we want to find all the diffeomorphisms $\phi : (\mathbb{R}^{1,n}, \tilde{\eta}) \to (\mathbb{R}^{1,n}, \eta)$ such that $\tilde{\eta} = \phi^* \eta$, i.e. in local coordinates

$$\tilde{\eta}_{ab} dx^a dx^b = (\eta_{ij} \circ \phi) d\phi^i d\phi^j. \tag{13}$$

Notice that (13) is equivalent to the following identity

$$\tilde{\eta}_{ab} = (\eta_{ij} \circ \phi) \frac{\partial \phi^i}{\partial x^a} \frac{\partial \phi^j}{\partial x^b}, \tag{14}$$

where we used the fact that $d\phi^i = \frac{\partial \phi^i}{\partial x^a} dx^a$. Let us now apply any partial derivative to this identity. Notice that since η_{ij} is a constant function, so is $\tilde{\eta}_{ab}$.

$$\begin{aligned} 0 =& \frac{\partial}{\partial x^k} \tilde{\eta}_{ab}(x^1, \dots, x^n) = \frac{\partial}{\partial x^k} \left((\eta_{ij} \circ \phi)(x^1, \dots, x^n) \frac{\partial \phi^i}{\partial x^a} \frac{\partial \phi^j}{\partial x^b} \right) \\ =& 2(\eta_{ij} \circ \phi) \frac{\partial^2 \phi^i}{\partial x^k \partial x^a} \frac{\partial \phi^j}{\partial x^b}. \end{aligned}$$

Since η_{ij} is a nonzero constant, $\frac{\partial \phi^i}{\partial x^b} \neq 0$ (because ϕ is a diffeomorphism), we require

$$\frac{\partial^2 \phi^i}{\partial x^k \partial x^a} = 0,$$

which implies

$$\phi^i(x^1, \dots, x^n) = L^i_a x^a + b^i,$$

where L^i_a and b^i are real numbers, where L^i_a represents a rotation and b^i a translation in $\mathbb{R}^{1,n} = \mathbb{R}^{n+1}$. In particular, it is possible to show that all these isometries form a group, called the *Poincarè group.*

4 The Lorentz Transformation as an Isometry

Consider the Lorentz transformation as derived in section 2. Furthermore, as one often does in theoretical physics, we consider natural units in which the speed of light is $c = 1$, such that the Lorentz factor becomes:

$$\gamma = \frac{1}{\sqrt{1 - v^2}}. \tag{15}$$

Then, the Lorentz transformation becomes

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx). \end{aligned} \tag{16}$$

We summarize this Lorentz transformation (16) by the linear transformation Φ defined as follows:

$$\begin{aligned} \Phi: \qquad & \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \\ & (t, x, y, z) \mapsto (t', x', y', z') = (\gamma(t - vx), y, z, \gamma(t - vx)). \end{aligned} \tag{17}$$

Now we verify that Φ is an isometry with respect to the Minkowski distance d, defined for two arbitrary points $(t_1, x_1, y_1, z_1) =: p_1 \in \mathbb{R}^4$, $(t_2, x_2, y_2, z_2) =: p_2 \in \mathbb{R}^4$ as

$$d((t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2)) := -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \tag{18}$$

i.e. we have to see that

$$d(\Phi(p_1), \Phi(p_2)) = d(p_1, p_2). \tag{19}$$

We compute the left hand side of equation (19):

$$\begin{aligned} & d(\Phi(p_1), \Phi(p_2)) \\ & = -(t_1' - t_2')^2 + (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2 \\ & = -\gamma^2 \left[(t_1 - t_2) - v(x_1 - x_2)\right]^2 + \gamma^2 \left[(x_1 - x_2) - v(t_1 - t_2)\right]^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ & = \gamma^2[-(t_1 - t_2)^2 - v^2(x_1 - x_2)^2 + 2v(t_1 - t_2)(x_1 - x_2) + (x_1 - x_2)^2 + v^2(t_1 - t_2)^2 \\ & \quad - 2v(t_1 - t_2)(x_1 - x_2)] + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ & = \frac{1}{1 - v^2} \left[(v^2 - 1)(t_1 - t_2)^2 + (1 - v^2)(x_1 - x_2)^2\right] + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ & = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ & = d(p_1, p_2), \end{aligned}$$

Where we applied the Lorentz transformation (16) and the definition of the Lorentz factor (15).

References

[1] Philosophiae naturalis principia mathematica - Isaac Newton; 1687

[2] Über die spezielle und die allgemeine Relativitätstheorie (Gemeinverständlich) - Albert Einstein; 1920

[3] Is the speed of light independent of the velocity of the source? - Brecher, K; 1977