Hermitian Young Projection Operators: Part I

Cian Roche, Bernanda Telalovic, Giuseppe Rizzello

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INTRODUCTION

This session will be largely repetition from Sessions 2 and 10, making sure we are comfortable with representations, the group algebra, birdtracks and Young operators. We will also introduce Hermiticity of Young operators, and in the next two sessions we will show how one can construct Hermitian Young operators from non-Hermitian ones.

https://youtu.be/S_0HGhraJb4 (2 min) (1)

RECAP

1. Representation

A representation (rep) Γ of a group G on a vector space V is a homomorphism (structure preserving map) from G to $\mathrm{GL}(V)$, the space of invertible linear maps on V.

https://youtu.be/q9uR--h9X68 (2 min) (2)

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2. Reducibility

- An invariant subspace $U \subseteq V$ is a subspace of V which is closed under the action of G through Γ.
- If V contains no nontrivial invariant subspace, then Γ is called an irreducible representation of G on V.

https://youtu.be/vMAYfb_B7PI (5 min) (3)

- Any representation of a finite group can be decomposed as a direct sum of these irreducible representations (Maschke's theorem).

https://youtu.be/07FpuDCB6-s (2 min) (4)

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3. Group Algebra

- We defined the group algebra $\mathscr{A}(G)$ of a group G as the vector space of complex linear combinations of the group elements.
- This space carried a special representation called the "regular representation" with the special property that the multiplicity of irrep Γ_μ was its dimension, d_μ.

$$\text{https://youtu.be/TEj2plp6NJg} \quad \text{(2 min)} \tag{5}$$

- The invariant subspaces of $\mathscr{A}(G)$ are labelled L_μ, and we can define projection operators $e_\mu \in \mathscr{A}(G)$ onto these subspaces, called a primitive idempotent of G.

The Primitive Idempotents of the Symmetric Group

For the Symmetric group S_n we can label a permutation $\sigma \in S_n$ by a standard Young Tableaux Θ with n boxes. For example if $n = 5$,

$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \longleftrightarrow (123)(45) \tag{6}$$

We define the row symmetriser s_Θ and column antisymmetriser a_Θ of a Tableaux Θ as

$$s_\Theta := \sum_{\text{rows}} (\text{ horizontal permutations }) \tag{7}$$

$$a_\Theta := \sum_{\text{columns}} \text{Sign}(\sigma)(\text{ vertical permutations }) \tag{8}$$

where $\text{Sign}(\sigma)$ is simply the sign of the vertical permutation in question. This is easiest to understand with an example:

$$\text{Symmetriser : https://youtu.be/9FzO4NRM4Ns} \quad \text{(3 min)} \tag{9}$$

$$\text{Antisymmetriser : https://youtu.be/VtyGulfwfgk} \quad \text{(3 min)} \tag{10}$$

Exercise 1: Explain why the normalisation of $1/n!$ in the definition of a birdtrack symmetriser is necessary by considering the following:

- Choose any two permutations $\sigma \in S_3$ and write them in birdtrack notation.
- Write down the symmetriser s_Θ for $\Theta = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ in birdtracks.
- Apply one of your chosen permutations to s_Θ and note the result, the apply the other permutation to s_Θ and compare what you got each time.

Hint: The symmetriser here is the sum of all $\sigma \in S_3$, and squaring the symmetriser would correspond to doing the above exercise for all 3! elements of S_3, and adding the results.

4. Young Operators

We now define the Young operator $Y_\Theta \in \mathscr{A}(G)$ corresponding to a tableaux Θ as

$$Y_\Theta := \frac{1}{|\Theta|} s_\Theta a_\Theta \tag{11}$$

where $|\Theta|$ is the product of the hook lengths of the Young diagram. One can show the following:

1. Y_Θ is a primitive idempotent of G, projecting onto the invariant subspace $L_\Theta \subset \mathscr{A}(G)$
2. Any permutation of the numbers in Θ will project onto the same subspace L_Θ, i.e. the invariant subspaces are labelled by diagrams of different shapes.

https://youtu.be/1PvEmknYMXY (5 min) (12)

Exercise 2: Write down the Young operator Y_Θ in birdtrack notation (including its prefactor involving hook lengths and factorials) for

$$\Theta = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array} \tag{13}$$

and convince yourself of why no symmetriser-antisymmetriser pair will ever be connected by two lines (which would make the diagram 0).

Note: We usually omit symmetrisers and antisymmetrisers over only one line.

Hermitian Young Operators

5. The Hermitian Conjugate

A Young operator Y_Θ projects onto the invariant subspace corresponding to Θ, however we may have multiple copies of this subspace, and our projection may not be orthogonal (i.e. Y_Θ may contain parts that map one copy of these invariant subspaces into another one).

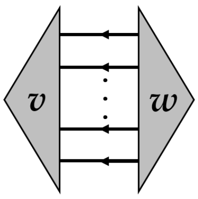
https://youtu.be/191WwSf1ZbE (2 min) (14)

Let V be any linear space. We can consider an operator $P : V^{\otimes n} \to V^{\otimes n}$. The Hermitian conjugate of the operator P, denoted $P^\dagger$, is defined by the relation

$$\langle v, Pw\rangle = \langle P^\dagger v, w\rangle \qquad \forall\, v, w \in V^{\otimes n} \tag{15}$$

Recall that in Session 2, we represented inner products as lines connecting two vectors, v, w, as

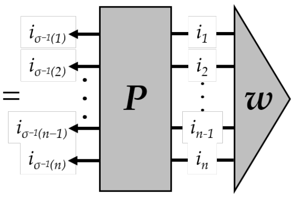
$$\langle v, w\rangle = \vec{v}^\dagger \vec{w} = v_{i_1 \dots i_n} w^{i_1 \dots i_n} =$$



where the notations are, from left to right, the inner product, vector notation, Einstein summation notation, and birdtrack notation, respectively.

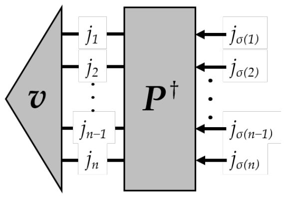
Consider the operator P now as a representation of S_n, and let us write $P(\sigma)w$ in birdtracks for $\sigma \in S_n$, permuting the indices just as in video 2:

$$P(\sigma)w = P(\sigma)\bigotimes_{k=1}^{n} w_{i_k} e^{i_k} = \bigotimes_{k=1}^{n} w_{i_{\sigma^{-1}(k)}} e^{i_k} =$$



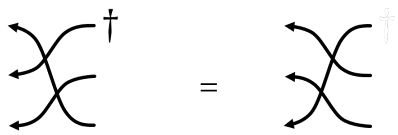
Now we want to see what the Hermitian conjugate of $P(\sigma)$ looks like. Formally, it is the map on the dual space $P^\dagger : \mathrm{Lin}(V^{\otimes n}, \mathbb{C}) \to \mathrm{Lin}(V^{\otimes n}, \mathbb{C})$. Let us write $P^\dagger(\sigma)v$ in birdtracks:

$$P^\dagger(\sigma)v = P^\dagger(\sigma)\bigotimes_{k=1}^{n} v_{j_k} e_{j_k} = \bigotimes_{k=1}^{n} v_{j_{\sigma(k)}} e_{j_k} =$$



https://youtu.be/cd0JEuklLDc (6 min) (16)

It turns out that the Hermitian conjugate of a general birdtrack operator is the same diagram, mirrored about the vertical axis (reversing the direction of the arrows).



Furthermore, an operator P is said to be Hermitian if $P = P^\dagger$. In general, any symmetric (about the vertical axis) birdtrack diagram will be manifestly Hermitian. However, this turns out to be a sufficient, but not necessary condition, as we will later see ways of simplifying diagrams.

https://youtu.be/32pObfLn2Is (2 min) (17)

The Hermiticity construction has the following consequence: The operators $P(\sigma)$ are unitary, that is $P(\sigma)P^\dagger(\sigma) = \mathbb{1}_n$.

https://youtu.be/5UkJPwQIIC8 (2 min) (18)

Exercise 3: For $\sigma = (123)(45)$:

1. Write the corresponding birdtrack diagram for this permutation.
2. Write out the flipped birdtrack diagram. Show that this birdtrack diagram is the same as that of σ^{-1} (therefore since flipping the diagram is taking the Hermitian conjugate, the diagram of part 2 is the Hermitian conjugate of the diagram in part 1).
3. (Optional) Show that they are unitary.

Exercise 4: For $\Theta = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$:

1. Write the corresponding birdtrack diagram for this permutation.
2. You should have one symmetriser and one antisymmetriser in your diagram. Separately for each of them, write out their components in full, then flip each of their components. What happens to symmetrisers and antisymmetrisers when they are flipped? Is your result true in general?